Nonlinear waves in cylindrically bounded magnetized plasmas

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Exact solutions are obtained for large amplitude waves propagating in cold magnetized cylindrical plasmas bounded by rigid dielectrics. Nonlinearly interacting bulk and surface waves are investigated.

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I. INTRODUCTION

The existence of exact solutions for finite amplitude waves propagating in plasmas bounded by dielectric containers has been pointed out recently [1–3]. These solutions satisfy the fluid equations for the plasma, Maxwell's equations for the electric and magnetic fields, as well as the appropriate boundary conditions. No approximation of any kind, such as perturbation or truncation, is invoked in obtaining the solutions. Such results are of special interest because, besides describing accurately the physical situation within the validity of the basic equations, they are helpful in verifying new approximations or numerical schemes in the study of nonlinear waves and instabilities.

Low-temperature plasmas are of interest [4–6] because of their relevance in modern technology, such as in laboratory plasma production and diagnostics, new sources of coherent radiation and particle beams, electronic and optical devices, as well as in machines for plasma-assisted material processing [7–13]. They differ from the hot, fusion-related plasmas in that they are confined by rigid containers instead of strong magnetic fields. The containers can be of either conducting or dielectric material. Thus natural oscillations can appear inside the plasma as bulk or volume waves as well as on the plasma-wall interface as surface waves. Since very large amplitude oscillations are often excited during the production or maintenace of low-temperature plasmas, it is important to understand the details of the nonlinear volume and surface modes and their interaction.

Often, external magnetic fields are used to control the parameters of low-temperature plasmas. In this paper, we investigate nonlinear plasma waves in and on the surface of a magnetized plasma bounded by a cylindrical dielectric. Exact solutions for nonlinear bulk and surface waves are obtained.

II. ELECTRON DYNAMICS

We consider here electrostatic waves in a cylindrical cold electron plasma in a positively charged background of immobile ions. The plasma is bounded at $r = R$ by a rigid dielectric of constant permittivity ϵ_d. An external magnetic field $B_0\hat{\mathbf{z}}$ along the cylinder axis is assumed to be present. The evolution of the electron density n is governed by the continuity equation

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0 \ . \tag{1}$$

The electron fluid velocity $\mathbf{v}$ satisfies the fluid momentum equation

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}_0) \ , \tag{2}$$

where $q = -e$ and m are the electron charge and mass, and $\mathbf{E}$ and $\mathbf{B}_0$ are the wave electric field and the external magnetic field. The plasma is assumed to be of low temperature such that the pressure effects can be neglected.

The approach to be used is similar to that of Lorenz [14], who investigated nonlinear waves and deterministic chaos in atmospheric physics by first separating the spatial variations from the temporal one. However, here no *ad hoc* truncation of the higher harmonics will be made.

Accordingly, for the spatial wave structure inside the plasma, we set

$$n = n(t), \tag{3}$$

$$v_r = (v_2 + v_1\cos 2\theta + v_4 \sin 2\theta) r/R \ , \tag{4}$$

$$v_\theta = -(v_3 - v_4\cos 2\theta + v_1 \sin 2\theta) r/R \ , \tag{5}$$

and

$$\varphi_{r<R} = (\varphi_c + \varphi_m \cos 2\theta + \varphi_n \sin 2\theta) r^2/R^2 - \varphi_c \ , \tag{6}$$

where cylindrical coordinates have been used. Here, v_1, v_2, v_3, v_4, φ_c, φ_m, and φ_n are functions of time only. We shall consider electrostatic oscillations, i.e., $\mathbf{E}_{r<R} = -\nabla\varphi_{r<R}$.

The Ansätze (3)–(6) have been found by trial and error, such that the time and space dependences of the resulting equations are separable. We have not been able to find a nontrivial simpler representation, but more complex representations involving higher spatial harmonics probably exist.

For the potential inside the dielectric, we assume

$$\varphi_{r>R} = (\varphi_m \cos 2\theta + \varphi_n \sin 2\theta)\, \frac{R^2}{r^2} \ , \qquad (7)$$

which satisfies the Laplace equation. Thus the corresponding electric field $\mathbf{E}_{r>R} = -\nabla\varphi_{r>R}$ in the dielectric decays as r^{-3} away from the cylinder.

The equations are completed by the Poisson equation

$$\nabla^2\varphi = -\frac{q}{\epsilon_0}(n - n_0) \ , \qquad (8)$$

where n_0 is the constant ion density. Note that the system does not have to be neutral.

The boundary condition, which is also a nonlinear relation, is

$$[qnv_r - \epsilon_0\partial_t\partial_r\varphi]_{r=R_{-0}} = [-\epsilon_0\epsilon_d\partial_t\partial_r\varphi]_{r=R_{+0}} \ , \qquad (9)$$

which represents the continuity of the current density across the interface between the plasma and the rigid dielectric.

Consistent with the cold-plasma approximation, the thickness (usually of the order of the Debye length) of the surface layer [13,16] at the interface is assumed to be smaller than any other characteristic dimension, in particular, the attenuation length of the surface-wave fields, and it is thus neglected. This thin layer also acts as a source and a sink for the plasma particles.

III. EVOLUTION EQUATIONS

As mentioned, the Ansätze (3) – (7) have been chosen such that the spatial and time dependences of the physical quantities are separable. In fact, substituting them into (1), (2), and (8), using the boundary condition (9), and equating terms with the same spatial dependence, one obtains

$$\dot{N} + 2NV_2 = 0 \ , \qquad (10)$$

$$\dot{V}_1 + 2V_1V_2 = \Omega V_4 - 2\phi_m \ , \qquad (11)$$

$$\dot{V}_2 + V_1^2 + V_2^2 - V_3^2 + V_4^2 = -2\phi_c - \Omega V_3 \ , \qquad (12)$$

$$\dot{V}_3 + 2V_2(V_3 - \Omega/2) = 0 \ , \qquad (13)$$

$$\dot{V}_4 + 2V_2V_4 = -2\phi_n - \Omega V_1 \ , \qquad (14)$$

$$\dot{\phi}_c = NV_2/2 \ , \qquad (15)$$

$$\dot{\phi}_m = NV_1/2(1 + \varepsilon_d) \ , \qquad (16)$$

and

$$\dot{\phi}_n = NV_4/2(1 + \varepsilon_d) \ , \qquad (17)$$

where we have defined $\Omega = \omega_c/\omega_p$, $N = n/n_0$, $V_j = v_j/R\omega_p$ $(j = 1,2,3,4)$, $\phi_k = \varepsilon_0\varphi_k/n_0qR^2$ $(k = c,m,n)$, and where the dot represents the derivative with respect to time, which is normalized by the inverse plasma frequency ω_p^{-1}. From (10) and (13), we note that $V_3 - \Omega/2 = \sqrt{\alpha}N$, where α is a positive constant.

By eliminating V_j, one can obtain a set of three second order ordinary differential equations,

$$N\ddot{N} - \frac{3}{2}\dot{N}^2 - \dot{\phi}_m^2 - \dot{\phi}_n^2 + 2\alpha N^4 = (1 - N)N^2 + \frac{1}{2}\Omega^2N^2 \ , \qquad (18)$$

$$N\ddot{\phi}_m - 2\dot{\phi}_m\dot{N} = \Omega N\dot{\phi}_n - \frac{\phi_m N^2}{1 + \varepsilon_d} \ , \qquad (19)$$

and

$$N\ddot{\phi}_n - 2\dot{\phi}_n\dot{N} = -\Omega N\dot{\phi}_m - \frac{\phi_n N^2}{1 + \varepsilon_d} \ , \qquad (20)$$

which is a set of coupled nonlinear ordinary differential equations describing the time evolution of waves with fixed spatial dependences.

IV. SOLUTIONS

Before proceeding with the numerical evaluation of (18) – (20), let us first briefly discuss the linear limit. One then finds that (18) yields the dispersion relation of the upper hybrid waves $\omega^2 = 2 - N_0 + \Omega^2$, where N_0 is the equilibrium density. On the other hand, Eqs. (19) and (20) lead to the dispersion relation $[\omega^2 - N_0/(1+\varepsilon_d)]^2 = \omega^2\Omega^2$, which describes the corresponding surface waves.

The coupled evolution equations (18) – (20) have been integrated numerically for different sets of initial values. For each case, we integrate up to more than μ (≈ 40) plasma periods, where μ is the square root of the ion-to-electron mass ratio. A typical result is shown in Fig. 1, where the initial values are $N = 1.1$, $\dot{N} = 0.03$, $\phi_m = 0.25$, $\dot{\phi}_m = 0.035$, $\phi_n = 0.1$, and $\dot{\phi}_n = 0.025$. Figure 2 shows the phase relations between the nonlinear modes. It is clear from the figures that the volume and surface waves are strongly coupled. In fact, the much lower frequency surface waves (ϕ_m, ϕ_n) are modulated by the volume wave (N) fluctuations. However, the coupling is not resonant. This occurs because the nonlinear terms in the evolution equations are such that frequency and wave vector matching among the three modes does not occur. For very large ε_d, of interest to the short-pulse technology, we also found very large amplitude solitonlike solutions for the volume waves. The corresponding magnitudes of the surface waves ϕ_m and ϕ_n, however, increase indefinitely in a steplike manner. This phenomenon is shown in Fig. 3, which is for the initial values $N = 0.9$, $\dot{N} = -0.1$, $\phi_m = 0.02$, $\dot{\phi}_m = -0.1$, $\phi_n = 0.03$, and $\dot{\phi}_n = 0.01$. Figure 4

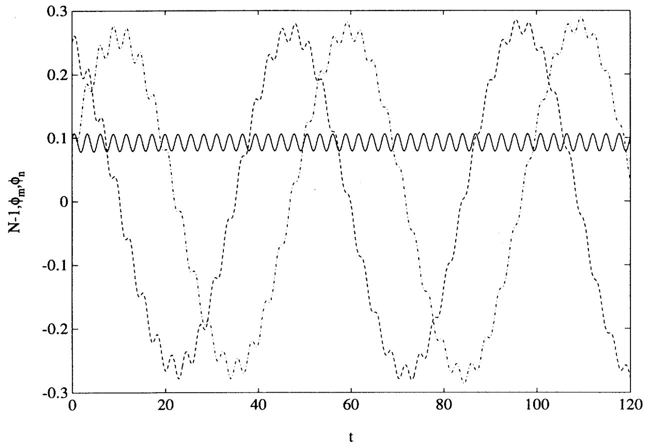

FIG. 1. $N-1$, ϕ_m and ϕ_n versus t. The solid, dashed, and dash-dotted lines are for $N-1$, ϕ_m and ϕ_n, respectively, and the parameters are Ω= 2, α= 0.8, and ε_d= 3.

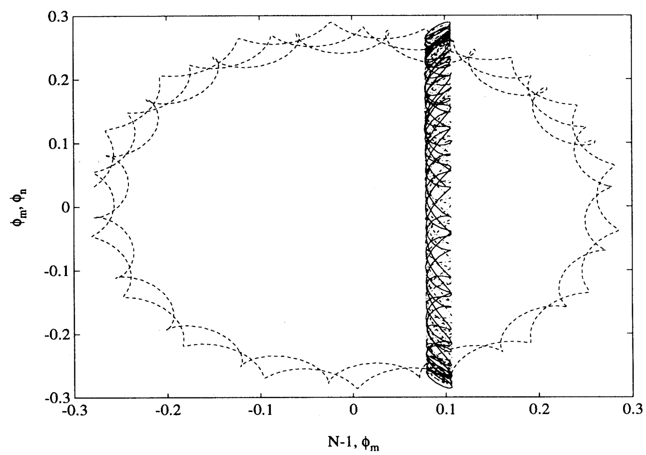

FIG. 2. Phase relations corresponding to Fig. 1. The solid, dashed, and dash-dotted lines are for ($N-1$, ϕ_m), (ϕ_m, ϕ_n), and ($N-1$, ϕ_n), respectively.

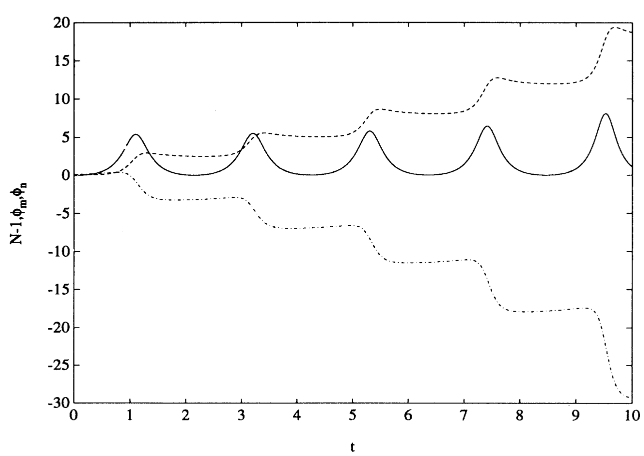

FIG. 3. Solitonlike solutions for $\varepsilon_d = 100$. The solid, dash, and dot lines are for $N-1$, ϕ_m and ϕ_n, respectively, and the parameters are $\Omega = 3$, $\alpha = 0.2$, and $\varepsilon_d = 100$.

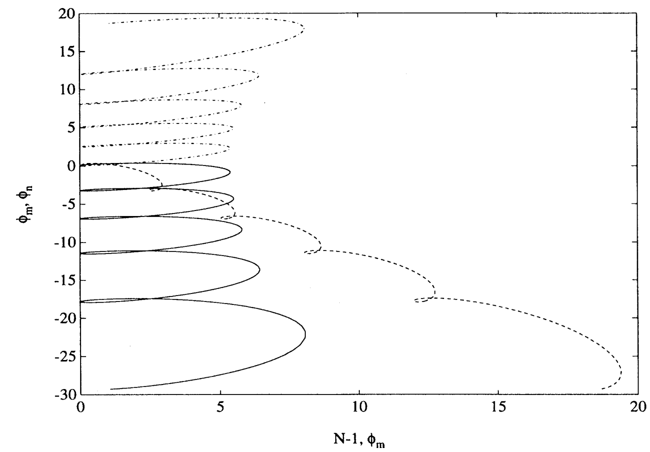

FIG. 4. The phase relations of the variables in Fig. 3. The solid, dashed, and dash-dotted lines denote ($N-1$, ϕ_m), (ϕ_m, ϕ_n), and ($N-1$, ϕ_n), respectively.

shows the corresponding phase relations. In general, we found that the results are not very sensitive to the initial conditions, but they are sensitive to the parameters Ω, α, and ε_d.

Although one cannot solve the system (18) – (20) analytically, in general, we found nevertheless after much trial and error a class of particular solutions which might be of certain interest. Accordingly, we make the Ansätze

$$N = (f_0 \sin \omega t + f_1)^{-1} , \tag{21}$$

$$\phi_m = (g_0 \sin \omega t + g_2 \cos \omega t + g_1) N , \tag{22}$$

$$\phi_n = (h_0 \sin \omega t + h_2 \cos \omega t + h_1) N , \tag{23}$$

where the $f_{0,1}$, $g_{0,1,2}$, $h_{0,1,2}$, and ω are constants. The latter can be determined if we insert the Ansätze (21)–(23) into (18) – (20).

As an example, for the case $\Omega \equiv 0$, we have $g_1 = g_2 = h_0 = h_1 = h_2 = 0$, $f_0^2 = f_1^2 + 2\alpha(2+\varepsilon_d)/\varepsilon_d$, $f_1 = 1/\omega^2(1+\varepsilon_d)$, $g_0^2 = (2-\omega^2)^3 f_0^2/2\omega^2$, and $\omega^2 = 1/(1+\varepsilon_d/2)$. In dimensional form, the latter becomes

$$\omega = \omega_p / \sqrt{1+\varepsilon_d/2} , \tag{24}$$

which characterizes a strongly nonlinear mode not pointed out previously. At a particular time, $t = t_\infty$, defined by $f_0 \sin \omega t_\infty + f_1 = 0$, both N and ϕ_m are singular and vary as $1/(t-t_\infty)^2$. A similar singularity can be found also in one-dimensional systems [15].

V. DISCUSSION

We have shown in this paper that exact nonlinear wave solutions can be constructed for a cylindrical magnetized low-temperature electron plasma bounded by a dielectric. Linearly, three distinct modes, one volume and two surface, exist in the system. Nonlinearly, these modes are selectively coupled, but not resonantly.

The present results may be relevant to surface-wave generated plasmas, control of plasmas for material processing, modulation of wave or beam pulses in fiber-optics communication, as well as for the verification of approximation and numerical schemes. Since the evolution equations are obtained without making use of any perturbations and truncations, when appropriate dissipation is added, the system may provide a mathematically exact model for investigating wave instabilities, saturation, and deterministic chaos [14]. It should be mentioned that here we have used the so-called sharp boundary model for a cold plasma. The model neglects the details of the transition, or boundary, layer [13,16] between the plasma and the dielectric. Thus, microscopic physical and chemical processes [13,16] which may occur in this layer have been neglected.

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